INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 11, 1033–1037 (1990)

A CHEBYSHEV MATRIX METHOD FOR THE SPATIAL MODES OF THE ORR–SOMMERFELD EQUATION

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SUMMARY

The Chebyshev matrix collocation method is applied to obtain the spatial modes of the Orr–Sommerfeld equation for Poiseuille flow and the Blasius boundary layer. The problem is linearized by the companion matrix technique. For semi-infinite domains a mapping transformation is used. The method can be easily adapted to problems with boundary conditions requiring different transformations.

KEY WORDS Chebyshev Spectral Nonlinear Eigenvalue

1. INTRODUCTION

The solution of non-linear eigenvalue problems can be obtained efficiently by shooting methods in which a good initial guess is vital for convergence. For problems in which a good initial guess is not available, the matrix method presents an attractive alternative. Bridges and Morris[1] developed a companion matrix method to linearize the spatial eigenvalue problem to study the stability of both channel and boundary layer flows. Their method incorporates the Chebyshev–tau method pioneered by Orszag[2] to discretize the governing Orr–Sommerfeld equation. Once the problem is linearized by this method, the entire spectrum can be obtained via the QZ or LZ algorithm. Recently, Khorrami *et al.*[3] used the Chebyshev matrix collocation method to study the temporal and spatial stability of swirling flows in enclosed domains. The matrix collocation method is easier than the tau method to formulate, does not require major modifications for each new velocity profile, and the boundary conditions do not pose a problem. Khorrami *et al.*[3] employed the companion matrix method of Bridges and Morris;[1] however, they did not present the eigenvector distributions corresponding to the least damped eigenvalue.

In this work the Chebyshev matrix collocation method[4–6] is combined with the companion matrix method to solve the non-linear spatial eigenvalue problem for channel and Blasius boundary layer flows. The semi-infinite domain of the Blasius flow requires a mapping transformation and presents a challenging problem for the method under consideration. In addition to the least damped eigenvalues, we also present the corresponding eigenfunction distributions which were obtained without recourse to a local solver.

2. SOLUTION PROCEDURE

The present numerical procedure uses the Chebyshev–Gauss–Lobatto points for the normal (y) direction discretization. For boundary layer calculations an exponential transformation[5] is used

0271–2091/90/151033–05$05.00

Received 16 May 1989
Revised 12 January 1990

to map the domain from $[0, \infty]$ to the half Chebyshev space $[1, 0]$. With this discretization the Orr–Sommerfeld equation can be written as

$$[(\mathbf{D}^2-\alpha^2\mathbf{I})^2 - \mathrm{i}Re\{(\alpha U\mathbf{I}-\omega\mathbf{I})(\mathbf{D}^2-\alpha\mathbf{I})-\alpha U''\mathbf{I}\}]\phi = 0, \tag{1}$$

$$\phi = \phi' = 0 \quad \text{at } y = \pm 1, \tag{1a}$$

$$\phi = \phi' = 0 \quad \text{at } y = 0,\ y \to \infty. \tag{1b}$$

Here ϕ is the eigenvector for the streamfunction, $\mathbf{D}$ represents the Chebyshev collocation matrix for the first derivative and $\mathbf{I}$ is the identity matrix. The elements of $\mathbf{D}$ are explicitly given by Canuto *et al.*[4] and the higher-order derivatives are simply obtained as powers of $\mathbf{D}$. In equation (1), α is the complex wave number, ω is the real, radian frequency and 'i' is the imaginary number; U and U'' represent the base flow velocity profile and its second derivative respectively. Re is the Reynolds number defined as $Re = U_0\delta_1/\nu$ for the boundary layer and $Re = U_1 h/\nu$ for channel flow. Here ν is the kinematic viscosity of the fluid, h is the half channel height and δ_1 is the displacement thickness. Velocity components are non-dimensionalized by the free stream velocity (U_0) and the maximum channel velocity (U_1) for the boundary layer and channel cases respectively.

For spatially-evolving problems, equation (1) is non-linear in the eigenvalue α and can be written as a polynomial in α in the following explicit form:

$$\mathbf{C}_4\alpha^4 + \mathbf{C}_3\alpha^3 + \mathbf{C}_2\alpha^2 + \mathbf{C}_1\alpha + \mathbf{C}_0 = 0 \tag{2}$$

with

$$\begin{aligned} \mathbf{C}_4 &= \mathbf{I}, \\ \mathbf{C}_3 &= \mathrm{i}ReU\mathbf{I}, \\ \mathbf{C}_2 &= -(\mathrm{i}\omega Re\mathbf{I} + 2\mathbf{D}^2), \\ \mathbf{C}_1 &= \mathrm{i}ReU''\mathbf{I} - \mathrm{i}ReU\mathbf{D}^2, \\ \mathbf{C}_0 &= \mathbf{D}^4 + \mathrm{i}Re\omega\mathbf{D}^2. \end{aligned} \tag{3}$$

In the above set of equations, all the bold-faced letters represent $(N+1) \times (N+1)$ matrices; the last four rows of these matrices are modified for the boundary conditions. Here N is the number of intervals in the domain. The boundary conditions are independent of the wave number and therefore are imposed only in $\mathbf{C}_0$, and the corresponding rows of the remaining matrices are set to zero. This implementation creates a singular $\mathbf{C}_4$ matrix resulting in infinite eigenvalues. In order to remedy the infinite eigenvalue problem and to increase solution accuracy by decreasing the number of arithmetic operations, the order of the matrices must be reduced. Since the zeros in the

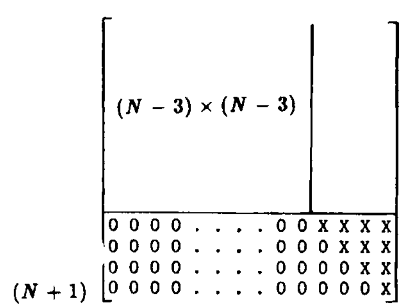

Figure 1. Structure of $\mathbf{C}_0$ after simple column operations

last four rows of the matrices (except $\mathbf{C}_0$) are to be preserved, the order of reduction is done by simple column operations. This results in a 4×4 upper triangular submatrix in the last four rows of the $\mathbf{C}_0$ matrix (Figure 1). If the boundary conditions are linearly independent, the order of the matrices is clearly reduced to $(N-3) \times (N-3)$ when four boundary conditions are eliminated. In these column operations, each column switching necessitates the switching of the corresponding rows of the calculated eigenvector. Therefore the original indices of the switched columns should be retained to decode the rows of the eigenvector matrix once it is calculated.

Following Bridges and Morris[1] and recalling that the eigenvalues of the companion matrix are the roots of the corresponding polynomial equation, a companion matrix for equation (2) can be written as

$$\left\{\begin{pmatrix} -\mathbf{C}_3 & -\mathbf{C}_2 & -\mathbf{C}_1 & -\mathbf{C}_0 \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{pmatrix} - \alpha \begin{pmatrix} \mathbf{C}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}\right\} \begin{Bmatrix} \alpha^3 \phi \\ \alpha^2 \phi \\ \alpha \phi \\ \phi \end{Bmatrix} = 0. \tag{4}$$

This equation represents a complex generalized eigenvalue problem and can be solved by the QZ algorithm. Note that the order of the above system is four times larger than the original reduced problem and the eigenfunctions can be directly obtained from the last quarter of the solution vector. The numerical calculations were performed on the VAX/VMS 8550 at the University of Colorado at Boulder and the CYBER 205 at NASA Langley Research Center. This was accomplished by the use of three IMSL subroutines, GVLCG, GVCCG and CXLZ.

3. RESULTS, DISCUSSION AND CONCLUSIONS

In this section a comparison of our results with those of Bridges and Morris[1] and Jordinson[7] is presented for the stability of channel and Blasius boundary layer flows respectively.

Channel flow stability

Here we concentrate on a test case given in Bridges and Morris[1] with $Re = 6000$ and $\omega = 0{\cdot}26$, which is linearly unstable for this flow. The first seven members of the eigenvalue spectrum are tabulated and compared with those of Bridges and Morris[1] in Table I. They employed the Chebyshev–tau expansion along with the matrix factorization technique using the Bernouilli iteration and solved the matrix eigenvalue problem with the QR algorithm. The present results

Table I. Comparison of the eigenvalue spectrum for spatial stability of plane Poiseuille flow ($Re = 6000$, $\omega = 0{\cdot}26$)

Mode	Bridges and Morris[1]	Present method	
		$N+1=41$	$N+1=51$
1	1·00047 − i0·00086	1·00046 − i0·00086	1·00047 − i0·00086
2	0·28323 + i0·02538	0·28323 + i0·02538	0·28323 + i0·02538
3	0·30165 + i0·04886	0·30165 + i0·04886	0·30165 + i0·04886
4	0·31976 + i0·07532	0·31976 + i0·07532	0·31976 + i0·07532
5	0·33745 + i0·10492	0·33748 + i0·10485	0·33745 + i0·10492
6	0·35456 + i0·13782	0·35664 + i0·13489	0·35456 + i0·13782
7	0·37090 + i0·17425		0·37089 + i0·17426

indicate that increasing the order of the Chebyshev polynomials results in noticeable improvement in accuracy, but machine and truncation errors impose a limit for the order of the polynomials that can be used; e.g. on the VAX/VMS 8550, increasing the order above 100 deteriorates the eigenvalue spectrum. This behaviour suggests that a trade-off exists between the order of the Chebyshev polynomials and the ability of the QZ algorithm to accurately solve large matrices.

Blasius boundary layer stability

Following Jordinson,[7] we performed three calculations corresponding to subcritical ($Re = 336$, $\omega = 0{\cdot}1297$), slightly unstable ($Re = 598$, $\omega = 0{\cdot}1201$) and unstable ($Re = 998$, $\omega = 0{\cdot}1122$) cases. The computed least damped eigenvalues are given in Table II, and the eigenfunction distributions shown in Figure 2 are in excellent agreement with those of Jordinson.[7] In Jordinson's work, a transformation of Numerov type resulting in fourth-order finite differences was used and a second-order iterative technique was applied to find the eigenvalues. Because of

Table II. Comparison of the least damped eigenvalue for spatial stability of the Blasius boundary layer (($N + 1$)-values are given in parentheses)

Re	ω	Jordinson[7]	Present method
336	0·1297	0·3084 + i0·0079 (81)	0·30864 + i0·00799 (46)
598	0·1201	0·3079 − i0·0019 (81)	0·30801 − i0·00184 (51)
998	0·1122	0·3086 − i0·0057 (81)	0·30870 − i0·00564 (51)

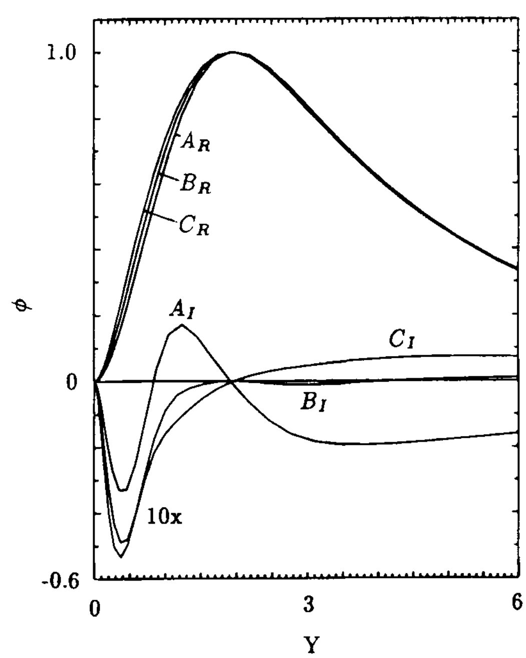

Figure 2. Real (R) and imaginary (I) parts of the eigenfunctions for: (A) $Re = 336$, $\omega = 0{\cdot}1297$; (B) $Re = 598$, $\omega = 0{\cdot}1201$; (C) $Re = 998$, $\omega = 0{\cdot}1122$ (Blasius boundary layer). Imaginary parts are ten times their actual magnitudes

the uniform mesh he utilized, the computational domain did not extend to infinity but was truncated at $6\delta_1$.

The results presented in this paper demonstrate the applicability of the Chebyshev matrix collocation method to non-linear eigenvalue problems including semi-infinite domains using the companion matrix approach. The advantage of the matrix collocation method in comparison with the tau method is the flexibility to use different co-ordinate transformations and to impose different boundary conditions with relative ease.

ACKNOWLEDGEMENT

This work was performed under Grant NAG-1-798 from NASA/Langley Research Center.

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